

The Laplace Transform in a Nutshell (CDT-9)

Luciano da Fontoura Costa
luciano@ifsc.usp.br

São Carlos Institute of Physics – DFCM/USP

June 4, 2019

Abstract

The Laplace transform is a powerful and versatile concept with broad applications in science and engineering. It allows a general means to represent and solve differential equations, while being related to other important mathematical concepts such as power series and the Fourier transform. In this text, an introduction to the Laplace transform is provided, with emphasis on graphic representations, also including examples of differential equation solutions related to pendulum oscillations and linear electric circuits.

‘Il tempo rimane abbastanza a lungo per coloro che lo utilizzano.’

Leonardo da Vinci.

1 Introduction

Scientific modeling is, to a great extent, founded on modeling (e.g. [1]) the phenomenon of interest in terms of differential equations, typically followed by finding or simulation their solution. Examples of this approach abound in physics, acoustics, mechanics, electricity, magnetism, thermal sciences, among many other areas. Therefore, it would be great if one could find a way to a more general approach to handling and solving differential equations, which would otherwise require specific integration techniques.

The good news is that such an approach exists: the *Laplace transform* (e.g. [2, 3, 4, 5, 6]). By transforming functions of a real variable t into a complex variable (e.g. [7]) s considering a basis of complex exponentials e^{-st} , this transform is capable of mapping equations and derivatives and integrals from *linear* differential equation into *polynomials* on s , therefore typically facilitating the respective solution.

The *one-sided* Laplace transform of a real or complex-valued function $f(t)$ of a real variable $t \geq 0$ is defined as:

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{\infty} f(t)e^{-st} dt. \quad (1)$$

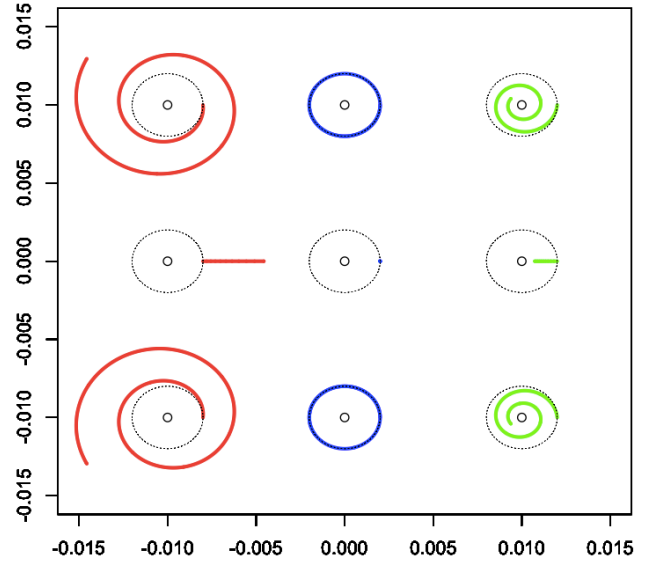


Figure 1: Some of the complex functions $\eta(s) = e^{-\sigma t} e^{-i\omega t}$ defined by some respective values $s = \sigma + i\omega$ in the complex plane. The functions in *blue* are respective to $\sigma = 0$, corresponding to the same basis of the Fourier transform. The functions in *green* have $\sigma < 0$, therefore exhibiting exponentially decaying behavior. The functions represented in *red* correspond to $\sigma > 0$ and, consequently, increase exponentially. Observe the symmetric ('mirrored') organization between the functions for $\omega > 0$ and respective $-\omega$ counterparts. The functions $\eta(s)$ are not shown to scale for better visualization's sake.

So, the Laplace transform maps the function $f(t)$, $t \geq 0$, into a complex function $F(s)$ of a complex variable

$s = \sigma + i\omega$, where σ and ω are real values. Keep in mind that this definition disregards the negative portion of the time domain of $f(t)$ even if it exists, which is related to the *causality* intrinsic to the one-sided Laplace transform.

Interesting insights about the essence of the Laplace transform can be inferred by considering the complex function $\eta(s) = e^{-st}$ of the complex variable s . Figure 1 illustrates this function for several values of s in the complex plane.

Only the complex semi-plane defined by $\sigma > 0$ is often considered in the Laplace transform, as several functions $f(t)$ converge in this region. Each instance of σ and ω defines a respective complex function $\eta(s) = e^{-(\sigma+i\omega)t} = e^{-\sigma t}e^{i\omega t}$, represented graphically in Figure 1. For $\sigma > 0$, we have that $\eta(s)$ decays steadily to zero for any value of ω , while oscillating according to ω . The limit situation where $\sigma \rightarrow 0$ is characterized by maintained oscillations.

Because the term $e^{-\sigma t}$ of $\eta(s)$ decays markedly with σ , the magnitude of the Laplace transform values obtained for increasing σ becomes smaller and smaller, in an exponential way, as illustrated in Figure 2, which shows $\eta(s)$ for $\sigma = 0.1$ and $\omega = 1$. The decaying exponential envelope is shown in orange.

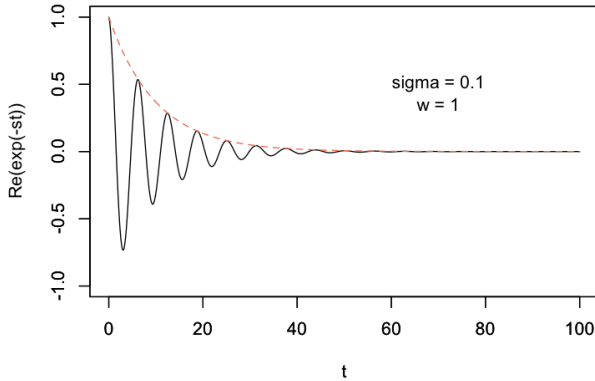


Figure 2: The exponentially decaying oscillating functions are the basis of the Laplace transform for $\sigma > 0$, where convergence is often observed. The orange curve corresponds to the exponential decaying function $e^{-\sigma t}$.

A *sufficient* condition for the Laplace transform of $f(t)$ to exist is that this function has *exponential order*, meaning that

$$|e^{-sf(t)}| < ce^{s(s-a)t},$$

for a real constant c and a real value $a \geq 0$ so that $\text{Re}(s) = \sigma > a$.

As an example, the Laplace transform of the *unit step function* ($u(t) = 1, t \geq 0$ and 0 otherwise) has as Laplace transform:

$$F(s) = \int_0^\infty e^{-st} dt = -\left[\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}. \quad (2)$$

This is valid only if $\text{Re}(s) > 0$ (i.e. $\sigma > 0$). It follows that the Laplace transform of a constant function $f(t) = c$ is c/s . We can also write

$$F(s) = \frac{1}{s} = \frac{1}{\sigma + i\omega} = \frac{\sigma - i\omega}{(\sigma + i\omega)(\sigma - i\omega)} = \frac{\sigma - i\omega}{\sigma^2 + \omega^2}$$

so that

$$\begin{aligned} \text{Re}(F(s)) &= \frac{\sigma}{\sigma^2 + \omega^2} \\ \text{Im}(F(s)) &= -\frac{\omega}{\sigma^2 + \omega^2} \end{aligned}$$

Figure 3 illustrates this Laplace transform graphically.

The unit step function allows us to express the important fact that

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{f(t)u(t)\}(s).$$

Another frequently used function is the *causal exponential* $f(t) = e^{-at}u(t)$, which has its Laplace transform calculated as

$$L(s) = \int_{s=0}^\infty e^{-at}e^{-st}dt = \left[\frac{1}{(s+a)}e^{(s-a)t}\right]_\infty^0 = \frac{1}{s+a},$$

for $\text{Re}(s) < -a$.

Table 1 lists several Laplace transforms of often considered functions, including the respective regions of convergence.

As it can be observed, the Laplace transforms tends to exhibit a ratio form, with the complex variable s in the denominator (i.e. $1/(s+a)$). This is also reflected in that visualizations of Laplace transforms tend to have an accentuated decay for increasing values of σ (i.e. along the real axis, as in Figure 3), though with varying number of peaks (e.g. the cosine and sine functions have two peaks in the respective Laplace transforms). Perhaps for this reason, Laplace transforms are not frequently visualized, though the position of roots and poles is often presented graphically in the complex plane.

The *two-sided* Laplace transform is defined as

$$\tilde{\mathcal{L}}\{f(t)\}(s) = \int_{-\infty}^\infty f(t)e^{-st}dt.$$

Observe that

$$\tilde{\mathcal{L}}\{f(t)u(t)\}(s) = \mathcal{L}\{f(t)\}(s).$$

The *Fourier transform*, a non-causal transform that is related to the *steady-state* regime, is related to the two-sided Laplace transform in the sense that $\mathcal{F}\{f(t)\} = F(s)|_{\sigma=0}$. We also have that

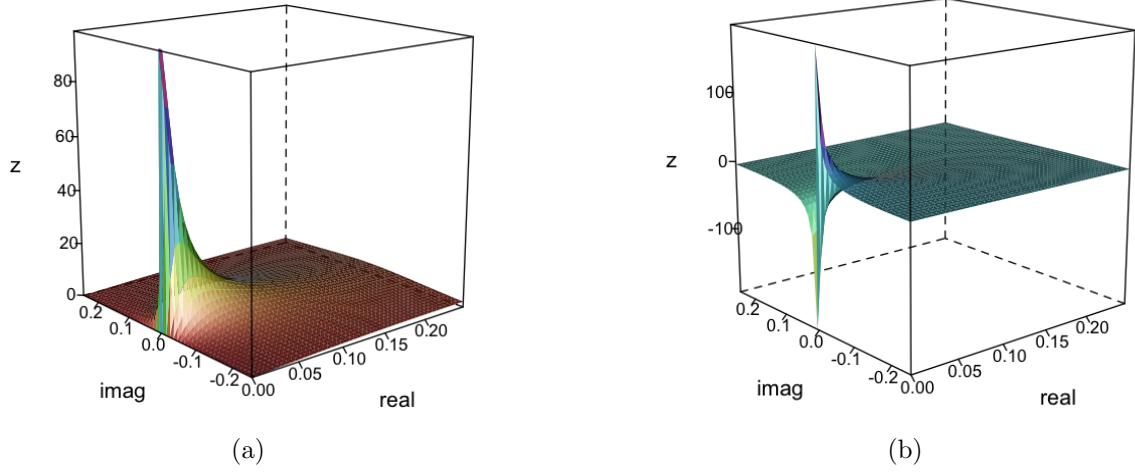


Figure 3: The real (a) and imaginary (b) parts of the Laplace transform of the unit step function $u(t)$, $\mathcal{L}\{u(t)\} = 1/s$.

$$\tilde{\mathcal{L}}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\alpha t}e^{-i\sigma t}dt = \mathcal{F}\{e^{-\alpha t}f(t)\}.$$

The Laplace transform is also related to other transforms (e.g. the Mellin and z -transform) and mathematical concepts (e.g. [5]) such as *statistical moments* and the *power series* of a discrete function $h[n]$, namely $\sum_{n=0}^{\infty} h[n]x^n$ (n is a positive integer and x is a real variable), corresponding to the limit situation when $n \rightarrow t$.

2 Properties of the Laplace Transform

The Laplace transform has several properties that are particularly useful. For instance, the *complex shifting* property says that

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad (3)$$

where a is any complex number.

We can combine Equations 2 and 3 to derive

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}.$$

In the particular case that $a = i\omega$, we have

$$\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s-i\omega}.$$

Now, we have that the cosine function can be expressed as $\cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/2$, so that

$$\begin{aligned} \mathcal{L}\{\cos(\omega t)\} &= \frac{1}{s-i\omega} + \frac{1}{s+i\omega} = \\ &= \frac{1}{2} \frac{s+i\omega + s-i\omega}{(s+i\omega)(s-i\omega)} = \frac{s}{s^2 + \omega^2}. \end{aligned}$$

In a similar way, considering that $\sin(\omega t) = (e^{i\omega t} - e^{-i\omega t})/(2i)$, we have that

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}.$$

We can also apply the complex shifting property to easily obtain the Laplace transform of $e^{-at}\cos(\omega t)$ as

$$\mathcal{L}\{e^{-at}\cos(\omega t)\} = \frac{\omega}{(s+a)^2 + \omega^2}.$$

Table 2 lists several properties of the Laplace transform.

3 The Inverse Laplace Transform

The inverse of the Laplace transform, namely $\mathcal{L}^{-1}\{F(s)\}$, so that $\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t)$, can be defined as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}ds,$$

where γ is a fixed real part of s , so that the integration proceeds along the respective defined straight line (parallel to the imaginary axis).

Because this integral can be difficult to be calculated, the inverse Laplace transform is often obtained by organizing $F(s)$ in a convenient way, so that tables of the Laplace transforms (such as Table 1) can be applied in order to derive the respective inverse. Examples of this procedure will be presented in the remainder of this text.

4 Ordinary Differential Equations

Let's consider a generic second order ordinary differential equation (e.g. [8])

$$a \frac{d^2 f(t)}{dt^2} + b \frac{df(t)}{dt} + c = g(t),$$

where the function $g(t)$ represents an external effect driving the dynamics of the modeled system. When $g(t) = 0$, we say that the equation is *homogeneous*.

The Laplace transform of this equation yields

$$a [s^2 F(s) - s f(0) - \dot{f}(0)] + b [s F(s) - f(0)] + c = G(s).$$

Thus, we can write

$$F(s) = \frac{(as + b)f(0) + a\dot{f}(0)}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

and it follows that

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A(s)}{Q(s)} + \frac{G(s)}{Q(s)}. \quad (4)$$

$Q(s)$ is called the *characteristic function* associated to the original differential equation. The term $\frac{A(s)}{Q(s)}$ can be understood as the Laplace transform of the *homogeneous* equation (i.e. excluding the driving function).

Given a ratio $R(s) = \frac{M(s)}{N(s)}$ of two complex function, the roots of $Q(s)$ are called the respective *poles* of R , while the roots of $M(s)$ are called simply the *roots* of R .

Let $Q(s)$ in Equation 4 have p_i all mutually distinct poles p_i . In this case, it can be shown that

$$\begin{aligned} \frac{P(s)}{Q(s)} &= \frac{P(s)}{(s - p_1)(s - p_2) \dots (s - p_M)} = \\ &= \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \dots + \frac{c_M}{s - p_M}, \end{aligned}$$

where the constants c_i can be obtained (e.g. [2]) as

$$c_i = \lim_{s \rightarrow p_i} \frac{(s - p_i)P(s)}{Q(s)}.$$

This procedure, called *partial fraction expansion*, will be illustrated in several of the following sections. Note that a different procedure applies when we have two or more identical poles.

5 Exponential Growth

One of the simplest ordinary differential equations is defined by the hypothesis that the rate of growth of a given function $f(t)$ is proportional to its current value (e.g. Malthus's population model), i.e.

$$\frac{d}{dt} f(t) = \alpha f(t).$$

The Laplace transform-based solution of this equation follows immediately as

$$\begin{aligned} sF(s) - f(0) - \alpha F(s) &= 0 \Rightarrow F(s) = \frac{f(0)}{s - \alpha} \Rightarrow \\ &\Rightarrow f(t) = f(0)e^{\alpha t}. \end{aligned} \quad (5)$$

6 The Harmonic Pendulum

The characterization of small oscillations of a pendulum can be approximated by the linearized (small oscillations) differential equation (e.g. [9]).

$$\ddot{\phi}(t) + c \phi(t) = g(t),$$

where $c = g/L$, and g is the gravity acceleration and L the length of the pendulum rod.

We can write

$$\begin{aligned} [s^2 \Phi(s) - s\phi(0) - \dot{\phi}(0)] + c\Phi(s) &= G(s) \Rightarrow \\ s^2 \Phi(s) + c\Phi(s) &= s\phi(0) + \dot{\phi}(0) + G(s) \Rightarrow \\ \Phi(s) &= \frac{s\phi(0) + \dot{\phi}(0)}{s^2 + c} + \frac{G(s)}{s^2 + c} = \\ &= \frac{P(s)}{Q(s)} = \frac{A(s)}{Q(s)} + \frac{G(s)}{Q(s)}. \end{aligned}$$

Let's consider that $g(t) = 0$, so that

$$\Phi(s) = \frac{A(s)}{Q(s)} = \frac{s\phi(0) + \dot{\phi}(0)}{s^2 + c}.$$

Assuming $\dot{\phi}(0) = 0$ and $\phi(0) = \phi_0$, we get

$$\Phi(s) = \phi_0 \frac{s}{s^2 + \omega_0^2},$$

where $\omega_0 = \sqrt{c} = \sqrt{g/L}$.

The solution of the original differential equation can now be found by taking the inverse Laplace transform (through Table 1), yielding

$$\phi(t) = \phi_0 \cos(\omega_0 t) = \phi_0 \cos\left(\sqrt{\frac{g}{L}} t\right).$$

7 The Driven Pendulum

Another interesting situation involving a linear approximation of a pendulum (small oscillations) is when an external force $g(t)$ is applied to drive the movement. This can be expressed as

$$\ddot{\phi}(t) + c \phi(t) = g(f).$$

As before, let's consider that $\dot{\phi}(0) = 0$ and $\phi(0) = \phi_0$, yielding

$$\Phi(s) = \frac{s\phi_0}{s^2 + c} + \frac{G(s)}{s^2 + c} = \frac{A(s)}{Q(s)} + \frac{G(s)}{Q(s)}.$$

We already obtained the inverse of $A(s)/Q(s)$, given as in Equation 6, which we will call $\phi_A(t) = \phi_0 \cos(\omega_0 t)$. For $g(t) = a \sin(\omega t)$, we have

$$\begin{aligned} \frac{G(s)}{Q(s)} &= \frac{\frac{a\omega}{s^2 + \omega^2}}{s^2 + c} = \frac{a\omega}{(s^2 + \omega^2)(s^2 + c)} = \\ &= \frac{a\omega}{(s - p_1)(s - p_2)(s - p_3)(s - p_4)} = \\ &= \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \frac{c_3}{s - p_3} + \frac{c_4}{s - p_4}. \end{aligned}$$

Partial fraction expansion yields

$$\frac{G(s)}{Q(s)} = \frac{a\omega}{(s + i\omega)(s - i\omega)(s + i\sqrt{c})(s - i\sqrt{c})}.$$

Since $(s^2 + \omega^2) = (s + i\omega)(s - i\omega)$, we immediately find that $p_1 = -i\omega$ and $p_2 = +i\omega$. The other two poles can be found by solving the quadratic equation $s^2 + c = 0$ as $p_3 = -i\sqrt{c}$ and $p_4 = i\sqrt{c}$. Therefore, it follows that

$$\phi(t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t} + c_3 e^{-i\sqrt{c}t} + c_4 e^{i\sqrt{c}t}.$$

Recall that $\omega_0 = \sqrt{c}$. The constants c_i can be calculated as

$$\begin{aligned} c_1 &= \frac{a}{2} \frac{i}{\omega_0^2 - \omega^2} = \\ c_2 &= -\frac{a}{2} \frac{i}{\omega_0^2 - \omega^2} = -c_1 \\ c_3 &= \frac{a}{2} \frac{i\omega}{\omega_0(\omega_0^2 - \omega^2)} \\ c_4 &= -\frac{a}{2} \frac{i\omega}{\omega_0(\omega_0^2 - \omega^2)} = -c_3. \end{aligned}$$

Now, we can write the solution respective to the term $\frac{G(s)}{Q(s)}$ as

$$\begin{aligned} \phi_S(t) &= c_1 e^{-i\omega t} - c_1 e^{i\omega t} + c_3 e^{-i\omega_0 t} - c_3 e^{i\omega_0 t} = \\ &= c_1 [e^{-i\omega t} - e^{i\omega t}] + c_3 [e^{-i\omega_0 t} - e^{i\omega_0 t}] = \\ &= \frac{ia}{2(\omega_0^2 - \omega^2)} \left[(e^{-i\omega t} - e^{i\omega t}) + \frac{i\omega}{\omega_0} (e^{-i\omega_0 t} - e^{i\omega_0 t}) \right] = \\ &= \frac{a}{\omega_0^2 - \omega^2} \sin(\omega t) + \frac{a\omega}{\omega_0(\omega_0^2 - \omega^2)} \sin(\omega_0 t). \end{aligned}$$

So, the full solution can be expressed as

$$\begin{aligned} \phi(t) &= \phi_T(t) + \phi_S(t) = \\ &= \phi_0 \cos(\omega_0 t) + \frac{a\omega}{\omega_0(\omega_0^2 - \omega^2)} \sin(\omega_0 t) + \frac{a}{\omega_0^2 - \omega^2} \sin(\omega t). \end{aligned}$$

Thus, we find that the driven pendulum will move as a superimposition of two oscillations with respective angular frequencies ω_0 and ω . Figure 4 illustrates the resulting movement $\phi(t)$ assuming $\phi_0 = \pi/200$, $g = 10$, $L = 0.1$ (so that $\omega_0 = 10 \text{ rad/s}$), $a = 0.2$, and $\omega = 12 \text{ rad/s}$.

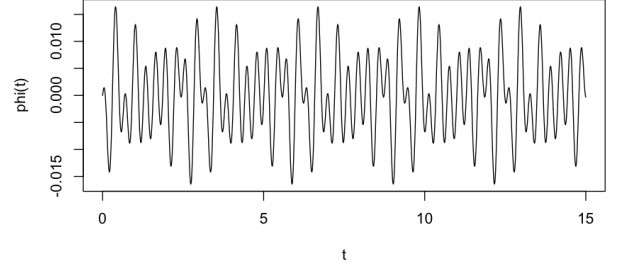


Figure 4: The oscillations of the small movement pendulum driven by a sinusoidal function, assuming $\phi_0 = \pi/200$, $g = 10$, $L = 0.1$ (so that $\omega_0 = 10 \text{ rad/s}$), $a = 0.2$, and $\omega = 12 \text{ rad/s}$.

8 The Damped Pendulum

Let's now consider that the pendulum movement is subjected to some damping (e.g. air resistance), which can be modeled as

$$\ddot{\phi}(t) = -\frac{g}{L}\phi(t) - \frac{k}{mL}\dot{\phi}(t).$$

Let's make $\alpha = \frac{g}{L}$ and $\beta = \frac{k}{mL}$, and write

$$\begin{aligned} \ddot{\phi}(t) + \alpha\phi(t) + \beta\dot{\phi}(t) &= 0 \Rightarrow \\ s^2\Phi(s) - s\phi(0) - \dot{\phi}(0) + \alpha\Phi(s) + \beta[s\Phi(s) - \phi(0)] &= 0 \Rightarrow \\ s^2\Phi(s) + \beta s\Phi(s) + \alpha\Phi(s) &= s\phi(0) + \dot{\phi}(0) + \beta\phi(0). \end{aligned}$$

Let's consider that $\dot{\phi}(0) = 0$ and $\phi(0) = \phi_0$, so that

$$\Phi(s) = \frac{(s + \beta)\phi_0}{s^2 + \beta s + \alpha}.$$

If $\beta^2 > 4\alpha$, we have two distinct real poles $p_1 = (-\beta + \sqrt{\beta^2 - 4\alpha})/2$ and $p_2 = (-\beta - \sqrt{\beta^2 - 4\alpha})/2$, and it follows that

$$\Phi(s) = \phi_0 \frac{s + \beta}{(s - p_1)(s - p_2)} = \phi_0 \left(\frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} \right),$$

and we find

$$\phi(t) = \phi_0 [c_1 e^{p_1 t} + c_2 e^{p_2 t}],$$

with $p_1 < 0$, $p_2 < 0$, and

$$c_1 = \lim_{s \rightarrow p_1} \frac{s + \beta}{s - p_2} = \frac{p_1 + \beta}{p_1 - p_2} = \frac{1}{2} \frac{\beta + \sqrt{\beta^2 - 4\alpha}}{\sqrt{\beta^2 - 4\alpha}}$$

$$c_2 = \lim_{s \rightarrow p_2} \frac{s + \beta}{s - p_1} = \frac{p_2 + \beta}{p_2 - p_1} = \frac{1}{2} \frac{\beta - \sqrt{\beta^2 - 4\alpha}}{\sqrt{\beta^2 - 4\alpha}}.$$

This parameter configuration is often called *over damped*, with the oscillations decreasing to zero exponentially, without oscillations, as implied by Equation 8.

Two other situations could be considered, namely: (i) $\beta^2 = 4\alpha$ (two identical real roots) and $\beta^2 < 4\alpha$ (two complex roots), which correspond respectively to the *critically damped* and *under damped* configurations. In the former case, the pendulum returns to the equilibrium as fast as possible, without oscillating and never reaching the equilibrium position. In the latter situation, the pendulum oscillates with decreasing amplitude as it returns to the equilibrium. Finding the respective solutions is left as an exercise.

9 RLC Series Circuit

Laplace transforms are widely used to solve linear circuits (e.g. [10]) involving resistors (R), capacitors (C) and inductors (L). In this section we apply the Laplace transform to solve a circuit composed of a voltage source $v(t)$ in series with a resistor, a capacitor and an inductor, as well as a switch that can be in open or closed position.

The circuit equation in the time domain is

$$R i(t) + \frac{1}{C} \int_0^t i(t) dt + v_C(0) + L \frac{d}{dt} i(t) = v(t).$$

which yields

$$R I(s) + \frac{1}{C} \frac{I(s)}{s} + \frac{v_C(0)}{s} + L [s I(s) - i(0)] = V(s).$$

Assuming that $i(0) = 0$ (i.e. the switch is open until $t = 0$, allowing the capacitor to maintain its initial condition $v_C(t)$ before that moment), we have

$$\left[R + \frac{1}{sC} + sL \right] I(s) = V(s) - \frac{v_C(0)}{s}.$$

Hence

$$I(s) = \frac{sV(s) - v_C(0)}{L \left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right)}.$$

The poles of the characteristic equation can be obtained as

$$s = -\frac{R}{2L} \pm i \sqrt{\frac{1}{LC} - \left(\frac{R}{2L} \right)^2} = -\sigma \pm i\omega.$$

Assuming $\frac{1}{LC} > \left(\frac{R}{2L} \right)^2$, it follows that

$$\begin{aligned} I(s) &= \frac{sV(s) - v_C(0)}{L(s + \sigma + i\omega)(s + \sigma - i\omega)} = \\ &= \frac{sV(s) - v_C(0)}{L[(s + \sigma)^2 + \omega^2]}. \end{aligned}$$

For $v(t) = Au(t)$, we have that $V(s) = A/s$, and

$$I(s) = \frac{A - v_C(0)}{L[(s + \sigma)^2 + \omega^2]}.$$

Thus we find the solution

$$i(t) = \frac{A - v_C(0)}{\omega L} e^{-\sigma t} \sin(\omega t).$$

Figure 5 illustrates this solution for $R = 0.1\omega$, $L = 1mH$, $C = 10mF$, $v_C(0) = 0.5V$, so that $i(0) = 0.5/R = 5A$.

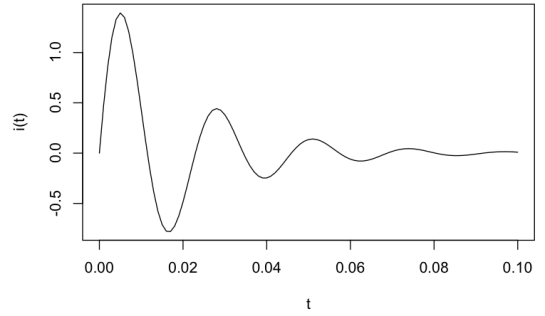


Figure 5: The solution $i(t)$ of the RLC series example for $R = 0.1\omega$, $L = 1mH$, $C = 10mF$, $v_C(0) = 0.5V$, so that $i(0) = 0.5/R = 5A$. Observe that the current $i(t)$ undergoes decreasing oscillations, converging steadily to zero.

10 Transient and Steady-State

Let's consider the previous RLC circuit, but now connected to a sinusoidal voltage source with fixed frequency, as it is often the case in electrical analysis. More specifically, consider that $g(t) = v(t) = \sin(\omega t)$, implying $V(s) = \frac{\omega}{s^2 + \omega^2}$.

We have from the previous section that

$$I(s) = \frac{\frac{s\omega}{s^2 + \omega^2} - v_C(0)}{L[(s + \sigma)^2 + \omega^2]}$$

which can be developed as

$$\begin{aligned} I(s) &= \frac{s\omega - s^2 v_C(0) - \omega^2 v_C(0)}{L(s^2 + \omega^2)[(s + \sigma)^2 + \omega^2]} = \\ &= \frac{1}{L} \frac{s\omega - s^2 v_C(0) - \omega^2 v_C(0)}{(s + i\omega)(s - i\omega)[(s + \sigma) + i\omega][(s + \sigma) - i\omega]}. \end{aligned}$$

Applying partial fraction expansion, we obtain

$$\frac{s\omega - s^2v_C(0) - \omega^2v_C(0)}{(s+i\omega)(s-i\omega)[(s+\sigma)+i\omega][(s+\sigma)-i\omega]} = \\ = \frac{C_1}{s+i\omega} + \frac{C_2}{s-i\omega} + \frac{C_3}{(s+\sigma)+i\omega} + \frac{C_4}{(s+\sigma)-i\omega}.$$

The respective solution is given as

$$i(t) = \frac{1}{L} [c_1 e^{-i\omega t} + c_2 e^{i\omega t} + c_3 e^{-\sigma t} e^{-i\omega t} + c_4 e^{-\sigma t} e^{i\omega t}],$$

For a sufficiently large time $t \rightarrow \infty$, the *transient* stage will have mostly vanished and the circuit will be in its *steady-state* regime, implying that

$$i_{steady}(t) = \frac{1}{L} [c_1 e^{-i\omega t} + c_2 e^{i\omega t}].$$

So, in the steady state, the current is a sinusoidal function with the same frequency as the voltage source. In this situation, the effect of the original charge in the capacitor will have almost completely vanished. This behavior allows us to consider the solution of steady-state circuits by using *phasors* (e.g. [11]).

11 Concluding Remarks

In this text, we provided an introduction to the Laplace transform and applications to solving differential equations. It should be kept in mind that this transform can have many other interesting applications, such as solving some types of partial differential equations, circuit networks, as well as being applied to many other problems.

In particular, the so-called *lumped* approach (e.g. [12]) allows the circuit analysis methodology illustrated here to be applied to other areas such as mechanics, acoustics, thermal sciences, magnetism, among many other possibilities.

Acknowledgments.

Luciano da F. Costa thanks CNPq (grant no. 307085/2018-0) for sponsorship. This work has benefited from FAPESP grant 15/22308-2.

Costa's Didactic Texts – CDTs

CDTs intend to be a halfway point between a formal scientific article and a dissemination text in the sense that they: (i) explain and illustrate concepts in a more informal, graphical and accessible way than the typical scientific article; and, at the same time, (ii) provide more in-depth mathematical developments than a more traditional dissemination work.

It is hoped that CDTs can also provide integration and new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

Though CDTs are intended primarily for those who have some preliminary experience in the covered concepts, they can also be useful as summary of main topics and concepts to be learnt by other readers interested in the respective CDT theme.

Each CDT focuses on a few interrelated concepts. Though attempting to be relatively self-contained, CDTs also aim at being relatively short. Links to related material are provided in order to complement the covered subjects.

The complete set of CDTs can be found at: <https://www.researchgate.net/project/Costas-Didactic-Texts-CDTs>.

References

- [1] L. da F. Costa. Modeling: The human approach to science. Researchgate, 2019. https://www.researchgate.net/publication/333389500_Modeling_The_Human_Approach_to_Science-CDT-8. Online; accessed 03-June-2019.
- [2] W. T. Thomson. *Laplace Transformation*. Prentice Hall, 1960.
- [3] Laplace transform. Wikipedia. https://en.wikipedia.org/wiki/Laplace_transform. Online; accessed 03-June-2019.
- [4] The Laplace transformation. Interactive Mathematics. <https://www.intmath.com/>

Table 1: Commonly used Laplace transforms.

$g(t)$	$F(s) = \mathcal{F}\{g(t)u(t)\}$	validity
$\delta(t)$	1	any $s \in \mathbb{C}$
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
a	$\frac{a}{s}$	$\text{Re}(s) > 0$
e^{-at}	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
$e^{-a t }$	$\frac{2a}{a^2 - s^2}$	$-a < \text{Re}(s) < a$
$(1 - e^{-at})$	$\frac{a}{s(s+a)}$	$\text{Re}(s) > 0$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\text{Re}(s) > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\text{Re}(s) > 0$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	$\text{Re}(s) > \omega $
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	$\text{Re}(s) > \omega $
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$	$\text{Re}(s) > -a$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$	$\text{Re}(s) > -a$
$t\cos(\omega t)$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$	$\text{Re}(s) > 0$
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	$\text{Re}(s) > 0$

Table 2: Some of the main Laplace transform properties.

$g(t)$	$G(s)$
$f(-t)$	$F(-s)$
$a f(t) + b h(t)$	$a F(s) + b H(s)$
$e^{at} f(t)$	$F(s - a)$
$f(t - a)u(t - a)$	$e^{-as} F(s)$
$f(at)$	$\frac{1}{ a } F\left(\frac{s}{a}\right)$
$tf(t)$	$-\frac{d}{ds} F(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(s) ds$
$\int_0^t f(t) dt$	$\frac{F(s)}{s}$
$\frac{d}{dt} f(t)$	$sF(s) - f(0)$
$\frac{d^2}{dt^2} f(t)$	$s^2 F(s) - sf(0) - \dot{f}(0)$
$x(t)\cos(\omega_0 t)$	$\frac{1}{2} [X(s - i\omega_0) + X(s + i\omega_0)]$
$x(t)\sin(\omega_0 t)$	$\frac{1}{2i} [X(s - i\omega_0) - X(s + i\omega_0)]$
$f(t) * h(t)$	$F(s)H(s)$
$f(t) h(t)$	$\frac{1}{2\pi i} F(s) * H(s)$
$f^*(t)$	$F^*(s^*)$
$\lim_{t \rightarrow 0} f(t) = f(0^+)$	$\lim_{s \rightarrow \infty} sF(s)$
$\lim_{t \rightarrow \infty} f(t) = f(\infty)$	$\lim_{s \rightarrow 0} sF(s)$

laplace-transformation/intro.php8. Online; accessed 03-June-2019.

[5] The Laplace transform. Wikipedia. https://en.wikipedia.org/wiki/Laplace_transform. Online; accessed 03-June-2019.

[6] E. Khutoryansky. Laplace transform explained and visualized intuitively. Youtube. <https://www.youtube.com/watch?v=6MXMDrs6ZmA>. Online; accessed 03-June-2019.

[7] Open University. An introduction to complex numbers. University Lecture, 2019. <https://www.open.edu/openlearn/science-maths-technology/mathematics-and-statistics/mathematics/introduction-complex-numbers/content-section-0?active-tab=description-tab>. Online; accessed 19-May-2019.

[8] R. K. Nagle, E. B. Saff, and A. D. Snider. *Fundamentals of Differential Equations*. Pearson, 2017.

[9] G. L. Baker and J. A. Blackburn. *The Pendulum: A Case Study in Physics*. Oxford Univ. Press, 2010.

[10] R. C. Dorf and J. A. Svoboda. *Introduction to Electric Circuits*. John Wiley and Sons, 2010.

[11] L. da F. Costa. Circuits, oscillations, and the Kuramoto model as visualized by phasors. Researchgate, 2019. https://www.researchgate.net/publication/333224636_Circuits_Oscillations_and_the_Kuramoto_Model_as_Visualized_by_Phasors_CDT-7. Online; accessed 03-June-2019.

[12] J. Voldman. Lumped-element modeling with equivalent circuits. MIT course. <https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-777j-design-and-fabrication-of-microelectromechanical-lecture-notes/07lecture08.pdf>. Online; accessed 03-June-2019.